Lead-Lag compensators: analytical and graphical design on the Nyquist plane

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I. INTRODUCTION

From classical control theory, it is known that the gain and phase margins (GPM) serve as important measures of the robustness of a dynamical system. The phase margin is related to the damping of the system, and therefore also serves as performance measurement [1]. Different methods can be found in the literature to satisfy GPM specifications [2]-[5]. However such solutions are usually obtained with trial-and-error methods. In 1998 a graphical method for the design of lead-lag compensators to satisfy GPM specifications was presented [2]. The recent literature shows a renewed interest in the design of classical controllers [6]-[9]. In this paper both a numerical and a graphical solution to the design of lead-lag compensators based on GPM specifications is proposed. A general structure for the lead-lag compensator with real and complex zeros and poles is used, relating it to the classical form with real zeros and poles. The paper is organized as follows: in Section II, the basic properties of this general form of lead-lag compensator are presented. In Section III the inversion formulae and their properties are described. In Section IV the design problems with constraints on the GPM are introduced, and their solutions are presented along with a graphical representation. Numerical examples and conclusions end the paper.

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II. LEAD-LAG COMPENSATORS: THE GENERAL STRUCTURE

Consider a lead-lag compensator described by the transfer function

\[ C(s) = \frac{s^2 + 2\gamma\delta\omega_n s + \omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}, \]  

where \( \gamma, \delta \) and \( \omega_n \) are real and positive. When \( \gamma\delta < 1 \) and/or \( \delta < 1 \) the zeros and/or the poles of the lead-lag compensator \( C(s) \) are complex conjugate with negative real part. The compensator \( C(s) \) has a unity static gain \( C(0) = 1 \) which does not change the static behavior of the controlled system. Notice that the compensator \( C(s) \) is written in a general form which encompasses the classical lead-lag form \( C_L(s) \) with real poles and real zeros:

\[ C_L(s) = \frac{(1 + \tau_1 s)(1 + \tau_2 s)}{(1 + \alpha \tau_1 s)(1 + \frac{\alpha}{\tau_2} s)} \]  

where \( 0 < \tau_1 < \tau_2 \) and \( 0 < \alpha < 1 \). The relations that link parameters in (1) and (2) are:

\[ \tau_1 = \frac{\gamma\delta - \sqrt{\gamma^2\delta^2 - 1}}{\omega_n}, \quad \tau_2 = \frac{\gamma\delta + \sqrt{\gamma^2\delta^2 - 1}}{\omega_n}, \quad \alpha = \frac{\delta - \sqrt{\delta^2 - 1}}{\gamma\delta - \sqrt{\gamma^2\delta^2 - 1}}. \]

The frequency response of compensator \( C(s) \) is

\[ C(j\omega) = \frac{1 + jX(\omega)}{1 + jY(\omega)}, \]

where

\[ X(\omega) = \frac{2\gamma\delta\omega \omega_n}{\omega_n^2 - \omega^2}, \quad Y(\omega) = \frac{2\delta\omega \omega_n}{\omega_n^2 - \omega^2}. \]

Since \( \gamma, \delta \) and \( \omega_n \) are assumed to be real and positive, \( X(\omega) \) and \( Y(\omega) \) are positive when \( \omega < \omega_n \) and negative when \( \omega > \omega_n \). The Nyquist diagram of \( C(j\omega) \) is a circle \( C(\gamma) \) with center \( C_0 = \frac{\gamma\delta}{2\omega_n} \) and radius \( R_0 = \frac{\gamma - \delta}{2\omega_n} \) (see Fig 1), that is

\[ C(\gamma) = C_0 + R_0 e^{j\theta} \quad \forall \theta \in [0, 2\pi]. \]

Proof: The distance \( d \) of the generic point \( C(j\omega) \) from the point \( C_0 \) is constant and equal to \( R_0 \),

\[ d^2 = |C(j\omega) - C_0|^2 = \frac{\omega_n^2 - \omega^2 + 2j\gamma\delta\omega_n\omega - \gamma + 1}{2(\omega_n^2 - \omega^2 + j2\delta\omega_n\omega)} \]

\[ = \frac{[(1 - \gamma)(\omega_n^2 - \omega^2) - j2\gamma\delta\omega_n\omega]^2}{2(\omega_n^2 - \omega^2 + j2\delta\omega_n\omega)} \]

\[ = \frac{(1 - \gamma)^2}{2(\omega_n^2 - \omega^2 + j2\delta\omega_n\omega)} \]

\[ = \frac{(\gamma - 1)^2}{2\omega_n^2} = R_0^2. \]