Systems and Control Theory

Master Degree Course in ELECTRONICS ENGINEERING

http://www.dii.unimore.it/~lbiagiotti/SystemsControlTheory.html

Some notes on identification of dynamic systems

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Identification of a dynamic system

- The procedure to obtain mathematical models of dynamic systems are usually calssified into **physical modeling** and **identification**

- **Physical modelling** is based on the partition of a system into subsystems and on their description by means of known laws. The model is then obtained by joining such relations into a whole (e.g. POG model)

- **Identification** (black-box) consists in the selection of a specific model in a specific class on the basis of observations performed on the system. The parameters of such models have not a physical meaning, but they are only a means for describing the input-output relation of the system

Least squares estimation

- Consider a **linear regression** model

$$y_k = \phi_1(x_k)\alpha_1 + \phi_2(x_k)\alpha_2 + \ldots + \phi_n(x_k)\alpha_n + \varepsilon_k$$

where
  - y is the observed variable
  - $\alpha_i, i = 1, \ldots, n$ are the unknown parameters
  - $\phi_i(\cdot), i = 1, \ldots, n$ are known functions of variable x
  - ε is the residual, that is the model error

The model is indexed by the integer variable k, which often assumes the meaning of a time

- **Problem**: given the pairs $\{y_k, x_k\}, k = 1, \ldots, N \ (N > n)$ determine the parameters $\alpha_i, i = 1, \ldots, n$ that minimize the cost function

$$V = \sum_{k=1}^{N} \varepsilon_k^2$$

Least squares estimation – cont'd

- With the notation

$$\mathbf{y} = [y_1, y_2, \ldots, y_N]^T, \varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N]^T, \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$$

$$\phi(k) = [\phi_1(x_k), \phi_2(x_k), \ldots, \phi_n(x_k)]^T \qquad \Phi = \begin{bmatrix} \phi^T(1) \\ \phi^T(2) \\ \cdots \\ \phi^T(N) \end{bmatrix}$$

- The cost function can be re-written as

$$\begin{cases} \min\limits_{\alpha} V(\alpha) = \min\limits_{\alpha} \varepsilon^T \varepsilon \\ \varepsilon = \mathbf{y} - \Phi\alpha \end{cases}$$

Least squares estimation – cont'd

- If the matrix $\Phi^T \Phi > 0$ then the function $V(\alpha)$ has a unique minimum given by

$$\hat{\alpha} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{y}$$

- The matrix $(\Phi^T \Phi)^{-1} \Phi^T$ is called pseudo-inverse of Φ (see command **pinv** in Matlab)

- The corresponding minimum value of $V(\alpha)$ is

$$\min_{\alpha} V(\alpha) = V(\hat{\alpha}) = \mathbf{y}^T \mathbf{y} - \mathbf{y}^T \Phi (\Phi^T \Phi)^{-1} \mathbf{y}$$

Least squares estimation – cont'd

- Demonstration

$$
\begin{aligned}
V(\alpha) &= \varepsilon^T \varepsilon \\
&= (\mathbf{y} - \Phi\alpha)^T (\mathbf{y} - \Phi\alpha) \\
&= \mathbf{y}^T\mathbf{y} - \mathbf{y}^T\Phi\alpha - \alpha^T\Phi^T\mathbf{y} + \alpha^T\Phi^T\Phi\alpha \\
&= \underbrace{[\mathbf{y}^T\mathbf{y} - \mathbf{y}^T\Phi(\Phi^T\Phi)^{-1}\mathbf{y}]}_{\text{Constant term}} + \underbrace{[\alpha - (\Phi^T\Phi)^{-1}\Phi^T\mathbf{y}]^T(\Phi^T\Phi)[\alpha - (\Phi^T\Phi)^{-1}\Phi^T\mathbf{y}]}_{\text{Positive definite quadratic form depending on } \alpha}
\end{aligned}
$$

Its minimum value is zero when $\alpha - (\Phi^T\Phi)^{-1}\Phi^T\mathbf{y} = 0$

$$\alpha = (\Phi^T\Phi)^{-1}\Phi^T\mathbf{y}$$

Least squares for linear dynamic systems

- Given a sample-data system

$$G(z) = \frac{b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} = \frac{Y(z)}{U(z)}$$

- In a generic instant k

$$\begin{aligned} y(k) &= -a_1 y(k-1) - a_2 y(k-2) - \cdots - a_n y(k-n) + \\ &\quad + b_1 u(k-1) + b_2 u(k-2) + \cdots + b_n u(k-n) + \varepsilon(k) \end{aligned}$$

where $a_i, b_i,\ i = 1, \ldots, n$ are unknown parameters and $\varepsilon(k)$ is the model residual

$$\varepsilon(k) = y(k) - \hat{y}(k, \alpha)$$

- This model is a particular case of a model called **ARX** (AutoRegresive model with eXternal input)

Least squares for linear dynamic systems

- Given N experiments, the following system of equations descends

$$
\begin{aligned}
y(n+1) &= -a_1\, y(n) - \cdots - a_n\, y(1) + b_1\, u(n) + \cdots + b_n\, u(1) + \varepsilon(n+1) \\
y(n+2) &= -a_1 y(n+1) - \cdots - a_n y(2) + b_1 u(n+1) + \cdots + b_n u(2) + \varepsilon(n+2) \\
&\vdots \\
y(N) &= -a_1 y(N-1) - \cdots - a_n y(N-n) + b_1 u(N-1) + \cdots + b_n u(N-n) + \varepsilon(N)
\end{aligned}
$$

- In a matrix form

$$\mathbf{y} = \Phi\alpha + \varepsilon$$

where

$$
\mathbf{y} = \begin{bmatrix} y(n+1) \\ y(n+2) \\ \vdots \\ y(N) \end{bmatrix}, \qquad
\alpha = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \qquad
\varepsilon = \begin{bmatrix} \varepsilon(n+1) \\ \varepsilon(n+2) \\ \vdots \\ \varepsilon(N) \end{bmatrix}
$$

Least squares for linear dynamic systems

and

$$\Phi = [\Phi_1 \;\; \Phi_2]$$

$$\Phi_1 = \begin{bmatrix} y(n) & y(n-1) & \cdots & y(1) \\ y(n+1) & y(n) & \cdots & y(2) \\ y(n+2) & y(n+1) & \cdots & y(3) \\ \vdots & & & \\ y(N-1) & y(N-2) & \cdots & y(N-n) \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} u(n) & u(n-1) & \cdots & u(1) \\ u(n+1) & u(n) & \cdots & u(2) \\ u(n+2) & u(n+1) & \cdots & u(3) \\ \vdots & & & \\ u(N-1) & u(N-2) & \cdots & u(N-n) \end{bmatrix}$$

- The solution to the least squares estimation problem is

$$\hat{\alpha} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{y}$$

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